

INDUCED CHARACTERS OF THE PROJECTIVE GENERAL LINEAR GROUP OVER A FINITE FIELD

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ABSTRACT. Using a general result of Lusztig, we find the decomposition into irreducibles of certain induced characters of the projective general linear group over a finite field of odd characteristic.

1. INTRODUCTION

Let q be a power of an **odd** prime, and \mathbb{F}_q the finite field with q elements. Fix an **even** integer $n \geq 2$. Consider the ordinary (complex) character theory of the projective general linear group $PGL_n(\mathbb{F}_q) = \{\bar{g} \mid g \in GL_n(\mathbb{F}_q)\}$, where \bar{g} denotes the coset of g relative to $Z(GL_n(\mathbb{F}_q)) \cong \mathbb{F}_q^\times$. The aim of this paper is to calculate the decomposition into irreducible characters of the following induced characters:

$$\text{Ind}_{PGSp_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1), \text{Ind}_{PGO_n^+(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1), \text{ and } \text{Ind}_{PGO_n^-(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1).$$

Specifically, for each of these induced characters I and for each irreducible character χ we will give a (manifestly nonnegative) combinatorial formula for the multiplicity $\langle \chi, I \rangle$, in the same spirit as the formulas given in [2] for the corresponding induced characters of $GL_n(\mathbb{F}_q)$.

Recall the definitions of these subgroups of $PGL_n(\mathbb{F}_q)$ (which are in fact only defined up to conjugacy). The group $PGL_n(\mathbb{F}_q)$ acts on the set

$$PGL_n(\mathbb{F}_q)^{\text{symm}} = \{\bar{h} \in PGL_n(\mathbb{F}_q) \mid \bar{h}^t = \bar{h}\} = \{\bar{h} \mid h \in GL_n(\mathbb{F}_q), h^t = \pm h\}$$

by the rule $\bar{g}.\bar{h} = \overline{ghg^t}$; we can think of $PGL_n(\mathbb{F}_q)^{\text{symm}}$ as the set of equivalence classes of nondegenerate symmetric and skew-symmetric bilinear forms on \mathbb{F}_q^n , where two forms are equivalent if they are scalar multiples of each other. The subgroup $PGSp_n(\mathbb{F}_q)$ (also called $PCSp_n(\mathbb{F}_q)$) is the stabilizer of an equivalence class of nondegenerate skew-symmetric forms. The subgroup $PGO_n^+(\mathbb{F}_q)$ is the stabilizer of an equivalence class of nondegenerate symmetric forms whose Witt index is $n/2$, and $PGO_n^-(\mathbb{F}_q)$ is the stabilizer of an equivalence class of nondegenerate symmetric forms whose Witt index is $n/2 - 1$. Since these three types of forms precisely give the three orbits of $PGL_n(\mathbb{F}_q)$ on $PGL_n(\mathbb{F}_q)^{\text{symm}}$, the character of the corresponding permutation representation is just the sum of the three induced characters we are calculating.

The reason for excluding the case when n is odd is that it is already known: $PGL_n(\mathbb{F}_q)$ acts transitively on $PGL_n(\mathbb{F}_q)^{\text{symm}}$, and the stabilizer $PGO_n(\mathbb{F}_q)$ is the image of $O_n(\mathbb{F}_q) = SO_n(\mathbb{F}_q) \times \{\pm 1\}$; in other words, $PGO_n(\mathbb{F}_q) = PSO_n(\mathbb{F}_q)$. Hence if n is odd,

$$(1.1) \quad \langle \chi, \text{Ind}_{PGO_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle = \langle \chi, \text{Ind}_{SO_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle = \langle \chi, \text{Ind}_{O_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle,$$

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where χ is regarded as an irreducible character of $GL_n(\mathbb{F}_q)$ via the canonical projection $GL_n(\mathbb{F}_q) \rightarrow PGL_n(\mathbb{F}_q)$. Thus [2, Theorem 4.1.1] gives a formula for this multiplicity.

By contrast, when n is even, the image in $PGL_n(\mathbb{F}_q)$ of the stabilizer of h in $GL_n(\mathbb{F}_q)$, namely

$$\{\bar{g} \in PGL_n(\mathbb{F}_q) \mid ghg^t = h\} = \{\bar{g} \in PGL_n(\mathbb{F}_q) \mid ghg^t = \alpha h, \exists \alpha \in (\mathbb{F}_q^\times)^2\},$$

is a subgroup of index 2 in the stabilizer of \bar{h} in $PGL_n(\mathbb{F}_q)$, namely

$$\{\bar{g} \in PGL_n(\mathbb{F}_q) \mid ghg^t = \alpha h, \exists \alpha \in \mathbb{F}_q^\times\}.$$

(The fact that α can take non-square values can be seen directly for $n = 2$ and then deduced for general even n .) If ω denotes, in each case, the homomorphism from the latter stabilizer to $\{\pm 1\}$ whose kernel is this subgroup, we have

$$(1.2) \quad \begin{aligned} \langle \chi, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle + \langle \chi, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(\omega) \rangle &= \langle \chi, \text{Ind}_{Sp_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle, \\ \langle \chi, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle + \langle \chi, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(\omega) \rangle &= \langle \chi, \text{Ind}_{O_n^+(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle, \\ \langle \chi, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle + \langle \chi, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(\omega) \rangle &= \langle \chi, \text{Ind}_{O_n^-(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle. \end{aligned}$$

So the formulas we are seeking cannot be deduced from the results in [2], but once found they can be combined with those results to give formulas for the induced characters $\text{Ind}(\omega)$ in the three cases. (If $q \equiv 1 \pmod{4}$ and $n \equiv 2 \pmod{4}$, this gives nothing substantially different, because ω is then the restriction of the unique nontrivial homomorphism $PGL_n(\mathbb{F}_q) \rightarrow \{\pm 1\}$.)

Since $\text{Ind}_{Sp_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)$ is multiplicity-free, $\text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1)$ must be also, so our formula merely specifies which constituents of the former are constituents of the latter; in the other cases, we have the inequalities

$$(1.3) \quad 0 \leq \langle \chi, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle \leq \langle \chi, \text{Ind}_{O_n^\epsilon(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle,$$

which will be obviously satisfied by our formulas.

Our approach will be the same as that of [2]: namely, we use Lusztig's formula [4, Theorem 3.3] for the inner product $\langle R_T^\lambda, I \rangle$, where R_T^λ is a Deligne-Lusztig character of $PGL_n(\mathbb{F}_q)$ and I is one of the above induced characters. Since the irreducible characters are linear combinations of the Deligne-Lusztig characters with known coefficients, all that remains is to interpret Lusztig's formula combinatorially and to manipulate it using various facts about characters of symmetric groups.

This paper does not attempt to be self-contained, and reference is freely made to the arguments and definitions of [2]. However, the more limited scope of the present inquiry (no unitary groups, no inner involutions) enables slight simplifications in the exposition and in the notation, which I hope will make the reader's job easier.

Since the complete formulas (Theorems 2.1 and 2.2 below) cannot be stated until we have introduced further notation, here are the answers in the special case of **unipotent** characters. Recall that the unipotent characters of $GL_n(\mathbb{F}_q)$, all of which factor through $PGL_n(\mathbb{F}_q)$, are parametrized by partitions of n , $\rho \mapsto \chi^\rho$; in our conventions, $\chi^{(n)}$ is the trivial character and $\chi^{(1^n)}$ is the Steinberg character. We have

$$(1.4) \quad \langle \chi^\rho, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle = \langle \chi^\rho, \text{Ind}_{Sp_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle = \begin{cases} 1, & \text{if } \rho \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

In other words, the unipotent constituents of $\text{Ind}_{Sp_n(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1)$ are all constituents of $\text{Ind}_{PGSp_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1)$ also. As for the other cases, recall from [2] that

$$(1.5) \quad \langle \chi^\rho, \text{Ind}_{O_n^\epsilon(\mathbb{F}_q)}^{GL_n(\mathbb{F}_q)}(1) \rangle = \frac{1}{2} \prod_i (m_i(\rho) + 1) + \begin{cases} \frac{1}{2}\epsilon, & \text{if } \rho' \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

Here $m_i(\rho)$ denotes the multiplicity of i as a part of ρ , and ρ' is the transpose partition (which is even if and only if $m_i(\rho)$ is even for all i). The new formula is:

$$(1.6) \quad \begin{aligned} \langle \chi^\rho, \text{Ind}_{PGO_n^\epsilon(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle &= \frac{1}{4} \prod_i (m_i(\rho) + 1) + \begin{cases} \frac{1}{2}\epsilon, & \text{if } \rho' \text{ is even} \\ 0, & \text{otherwise} \end{cases} \\ &+ \begin{cases} \frac{(-1)^{\frac{1}{2}\ell(\rho)_1}}{4} \prod_i (m_{2i}(\rho) + 1), & \text{if } 2 \mid m_{2i+1}(\rho), \forall i \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Here $\ell(\rho)_1 = \sum_i m_{2i+1}(\rho)$ is the number of odd parts of ρ .

Example. The multiplicities of the unipotent characters of $PGL_4(\mathbb{F}_q)$ in the above induced characters are given in the following table:

	(4)	(31)	(2 ²)	(21 ²)	(1 ⁴)
$\text{Ind}_{PGSp_4(\mathbb{F}_q)}^{PGL_4(\mathbb{F}_q)}(1)$	1	0	1	0	0
$\text{Ind}_{PGO_4^+(\mathbb{F}_q)}^{PGL_4(\mathbb{F}_q)}(1)$	1	1	2	1	2
$\text{Ind}_{PGO_4^-(\mathbb{F}_q)}^{PGL_4(\mathbb{F}_q)}(1)$	1	1	1	1	1

See (2.7) below for the complete decomposition in the case $n = 2$.

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2. NOTATION AND STATEMENT OF RESULTS

For general combinatorial notation, we follow [5]. For instance, we write $\mu \vdash m$ to mean that μ is a partition of m ; the size m can then be written $|\mu|$. The nonzero parts of μ are $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{\ell(\mu)}$. The transpose partition is μ' . For $i \in \mathbb{Z}^+$, the multiplicity of i as a part of μ is written $m_i(\mu)$. We say that μ is even if all its parts are even, or equivalently if $2 \mid m_i(\mu')$, $\forall i$.

We write w_μ for a permutation in $S_{|\mu|}$ whose cycle-type is μ (determined up to conjugacy), $\epsilon_\mu \in \{\pm 1\}$ for its sign, and z_μ for the order of its centralizer in $S_{|\mu|}$.

As well as $\ell(\mu)$ for the length of μ , we will use the notations $\ell(\mu)_0$ and $\ell(\mu)_1$ for the number of even and odd (nonzero) parts of μ respectively. Note that $\epsilon_\mu = (-1)^{\ell(\mu)_0}$. On occasion we will need to further analyse $\ell(\mu)_0$ into $\ell(\mu)_{0 \bmod 4}$ and $\ell(\mu)_{2 \bmod 4}$.

For any finite group Γ , $\widehat{\Gamma}$ denotes the set of irreducible (complex) characters of Γ . We have $\widehat{S_m} = \{\chi^\rho \mid \rho \vdash m\}$, where χ^ρ is as in [5, I.7]. Write χ_μ^ρ for the value of χ^ρ at an element of cycle-type μ , so that $\chi_\mu^{(m)} = 1$ and $\chi_\mu^{\rho'} = \epsilon_\mu \chi_\mu^\rho$.

Throughout the paper, we fix an algebraic closure k of \mathbb{F}_q , a k -vector space V of dimension n (our fixed **even** positive integer) and a Frobenius map $F : V \rightarrow V$ relative to \mathbb{F}_q . Let $G = GL(V)$, $\overline{G} = PGL(V)$, and write F also for the induced Frobenius maps on these groups. Thus $G^F \cong GL_n(\mathbb{F}_q)$ and $\overline{G}^F \cong PGL_n(\mathbb{F}_q)$.

For any subgroup H of G , we write \overline{H} for the image of H under the canonical projection $G \rightarrow \overline{G}$; in other words, $\overline{H} = HZ/Z$ where $Z = Z(G) \cong k^\times$ is the kernel. Because the Lang map of Z is surjective, we have a short exact sequence $1 \rightarrow Z^F \rightarrow (HZ)^F \rightarrow \overline{H}^F \rightarrow 1$. (But note that $(HZ)^F$ is in general larger than $H^F Z^F$, so $H^F \rightarrow \overline{H}^F$ is not necessarily surjective.)

As in [5, Chapter IV], we need to define the ‘dual group’ of k^\times : consider the system of maps $\widehat{\mathbb{F}_{q^e}^\times} \rightarrow \widehat{\mathbb{F}_{q^{e'}}^\times}$ for $e \mid e'$ (the transpose of the norm map), and its direct limit $L = \varinjlim \widehat{\mathbb{F}_{q^e}^\times}$. Let σ denote the q -th power map on both k^\times and L , so that $(k^\times)^{\sigma^e} \cong \mathbb{F}_{q^e}^\times$, $L^{\sigma^e} \cong \widehat{\mathbb{F}_{q^e}^\times}$ for all $e \geq 1$. Write $\langle \cdot, \cdot \rangle_e : \mathbb{F}_{q^e}^\times \times L^{\sigma^e} \rightarrow \mathbb{C}^\times$ for the canonical pairing, which is ‘bimultiplicative’ in the sense that

$$\langle \alpha_1 \alpha_2, \xi \rangle_e = \langle \alpha_1, \xi \rangle_e \langle \alpha_2, \xi \rangle_e, \quad \langle \alpha, \xi_1 \xi_2 \rangle_e = \langle \alpha, \xi_1 \rangle_e \langle \alpha, \xi_2 \rangle_e,$$

for all $\alpha, \alpha_1, \alpha_2 \in \mathbb{F}_{q^e}^\times$, $\xi, \xi_1, \xi_2 \in L^{\sigma^e}$. We will use without comment the following rules relating the pairings $\langle \cdot, \cdot \rangle_d$ and $\langle \cdot, \cdot \rangle_e$ when $d \mid e$:

- if $\alpha \in \mathbb{F}_{q^d}^\times$ and $\xi \in L^{\sigma^e}$, then

$$\langle \alpha, \xi \rangle_e = \langle \alpha, \xi^{1+q^d+q^{2d}+\dots+q^{e-d}} \rangle_d;$$

- if $\alpha \in \mathbb{F}_{q^e}^\times$ and $\xi \in L^{\sigma^d}$, then

$$\langle \alpha, \xi \rangle_e = \langle \alpha^{1+q^d+q^{2d}+\dots+q^{e-d}}, \xi \rangle_d;$$

- if $\alpha \in \mathbb{F}_{q^d}^\times$ and $\xi \in L^{\sigma^d}$, then

$$\langle \alpha, \xi \rangle_e = \langle \alpha, \xi \rangle_d^{e/d}.$$

When $e = 1$ we omit the subscript in $\langle \cdot, \cdot \rangle_e$ and write simply $\langle \cdot, \cdot \rangle$.

We fix some set of representatives for the orbits of the group $\langle \sigma \rangle$ generated by σ on L , and call it $\langle \sigma \rangle \backslash L$. For $\xi \in \langle \sigma \rangle \backslash L$, let $m_\xi = |\langle \sigma \rangle \cdot \xi|$, in other words the smallest $e \geq 1$ such that $\xi^{q^e} = \xi$. Let $N(\xi) = \xi^{1+q+q^2+\dots+q^{m_\xi-1}} \in L^\sigma$. Let $d_\xi = \langle -1, N(\xi) \rangle = \langle -1, \xi \rangle_{m_\xi}$, which is 1 if $L^{\sigma^{m_\xi}}$ contains square roots of ξ , and -1 if it does not. (Calling this sign d_ξ is the piece of notation in [2] I most regret, but it would be too confusing to change now.) Let η denote the unique element of L^σ with order 2; note that $d_\eta = 1$ if and only if $q \equiv 1 \pmod{4}$.

Let $\widehat{\mathcal{P}}_n$ be the set of collections of partitions $\underline{\nu} = (\nu_\xi)_{\xi \in L}$, almost all zero, such that $\sum_{\xi \in L} |\nu_\xi| = n$. Let $\widehat{\mathcal{P}}_n^\sigma$ be the subset of $\widehat{\mathcal{P}}_n$ of all $\underline{\nu}$ such that $\nu_{\sigma(\xi)} = \nu_\xi$ for all ξ . Note that for $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$,

$$(2.1) \quad \sum_{\xi \in \langle \sigma \rangle \backslash L} m_\xi |\nu_\xi| = n.$$

For $\underline{\nu}, \underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$, we write $|\underline{\nu}| = |\underline{\rho}|$ to mean that $|\nu_\xi| = |\rho_\xi|$ for all ξ . For $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$ we define

$$\Pi(\underline{\nu}) := \prod_{\xi \in L} \xi^{|\nu_\xi|} = \prod_{\xi \in \langle \sigma \rangle \backslash L} N(\xi)^{|\nu_\xi|} \in L^\sigma,$$

and let $\widehat{\mathbb{P}}_n^\sigma$ denote the subset of $\widehat{\mathcal{P}}_n^\sigma$ consisting of all $\underline{\nu}$ such that $\Pi(\underline{\nu}) = 1$. Finally, if $\underline{\nu} \in \widehat{\mathbb{P}}_n^\sigma$ is such that $m_\xi|\nu_\xi|$ is always even, we define

$$\Phi(\underline{\nu}) := \prod_{\xi \in \langle \sigma \rangle \setminus L} \langle \sqrt{\beta}, \xi \rangle_{m_\xi|\nu_\xi|}$$

for some $\sqrt{\beta} \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ whose square β lies in \mathbb{F}_q . We have $\Phi(\underline{\nu}) \in \{\pm 1\}$ since its square is $\langle \beta, \Pi(\underline{\nu}) \rangle = 1$. Moreover, $\Phi(\underline{\nu})$ is independent of the element $\sqrt{\beta}$ used to define it, because multiplying $\sqrt{\beta}$ by $\gamma \in \mathbb{F}_q^\times$ multiplies $\Phi(\underline{\nu})$ by $\langle \gamma, \Pi(\underline{\nu}) \rangle = 1$.

Now there is a well-known bijection between $\widehat{\mathcal{P}}_n^\sigma$ and the set of G^F -orbits of pairs (T, λ) where T is an F -stable maximal torus of G and $\lambda \in \widehat{T^F}$. If (T, λ) is in the orbit corresponding to $\underline{\nu}$ under this bijection, we have the following properties:

- the fixed lines of T can be labelled

$$(2.2) \quad \{L_{(\xi, j, i)} \mid \xi \in \langle \sigma \rangle \setminus L, 1 \leq j \leq \ell(\nu_\xi), i \in \mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z}\}$$

so that if $t \in T$ has eigenvalue $\alpha_{(\xi, j, i)}$ on each $L_{(\xi, j, i)}$, $F(t)$ has eigenvalue $\alpha_{(\xi, j, i-1)}^q$ on each $L_{(\xi, j, i)}$;

- consequently,

$$(2.3) \quad T^F \cong \prod_{\xi \in \langle \sigma \rangle \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} \mathbb{F}_{q^{m_\xi(\nu_\xi)_j}}^\times$$

via the map sending t to the collection $(\alpha_{(\xi, j, 1)})$;

- for $t \in T^F$ as above,

$$(2.4) \quad \lambda(t) = \prod_{\xi \in \langle \sigma \rangle \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} \langle \alpha_{(\xi, j, 1)}, \xi \rangle_{m_\xi(\nu_\xi)_j}.$$

From (2.4) it is clear that λ is trivial on Z^F if and only if $\Pi(\underline{\nu}) = 1$. Hence we obtain a bijection between $\widehat{\mathbb{P}}_n^\sigma$ and the set of \overline{G}^F -orbits of pairs (\overline{T}, λ) where \overline{T} is an F -stable maximal torus of \overline{G} and $\lambda \in \widehat{\overline{T}^F}$.

For $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$, let $B_{\underline{\nu}}$ be the corresponding **basic character** of G^F , defined by Green in [1]. As proved by Lusztig in [3], this coincides with the character of the Deligne-Lusztig virtual representation R_T^λ for (T, λ) in the corresponding G^F -orbit. From either point of view it is clear that $B_{\underline{\nu}}(zg) = \langle z, \Pi(\underline{\nu}) \rangle B_{\underline{\nu}}(g)$ for all $g \in G^F$ and $z \in Z^F$. For $\underline{\nu} \in \widehat{\mathbb{P}}_n^\sigma$, $B_{\underline{\nu}}$ may be viewed as a character of \overline{G}^F ; it is again the character of the corresponding Deligne-Lusztig representation $R_{\overline{T}}^\lambda$.

Green's main result on the character theory of $GL_n(\mathbb{F}_q)$ states that for any $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$,

$$(2.5) \quad \chi^\underline{\rho} := (-1)^{n+\sum_{\xi \in \langle \sigma \rangle \setminus L} |\rho_\xi|} \sum_{\substack{\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma \\ |\underline{\nu}| = |\underline{\rho}|}} \left(\prod_{\xi \in \langle \sigma \rangle \setminus L} (z_{\nu_\xi})^{-1} \chi_{\nu_\xi}^{\rho_\xi} \right) B_{\underline{\nu}}$$

is an irreducible character of G^F , and all irreducible characters arise in this way for unique $\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma$. (Note that our parametrization differs from that in [5, Chapter IV] by transposing all partitions.) Inverting the transition matrix, we have that for

any $\underline{\nu} \in \widehat{\mathcal{P}}_n^\sigma$,

$$(2.6) \quad B_{\underline{\nu}} = (-1)^{n + \sum_{\xi \in \langle \sigma \rangle \setminus L} |\nu_\xi|} \sum_{\substack{\underline{\rho} \in \widehat{\mathcal{P}}_n^\sigma \\ |\underline{\rho}| = |\underline{\nu}|}} \left(\prod_{\xi \in \langle \sigma \rangle \setminus L} \chi_{\nu_\xi}^{\rho_\xi} \right) \chi_{\underline{\nu}}^{\underline{\rho}}.$$

Moreover, $\chi_{\underline{\nu}}^{\underline{\rho}}(zg) = \langle z, \Pi(\underline{\rho}) \rangle \chi_{\underline{\nu}}^{\underline{\rho}}(g)$ for all $g \in G^F$ and $z \in Z^F$, so the characters which descend to irreducible characters of \overline{G}^F are exactly $\{\chi_{\underline{\nu}}^{\underline{\rho}} \mid \underline{\rho} \in \widehat{\mathbb{P}}_n^\sigma\}$. This includes the unipotent irreducible characters referred to in the introduction, which are those $\chi_{\underline{\nu}}^{\underline{\rho}}$ for which $\rho_\xi = 0$ unless $\xi = 1$. (In the introduction we parametrized these by $\rho = \rho_1$.)

With this notation we can state the results of this paper, which should be compared with [2, Theorem 2.1.1] and [2, Theorem 4.2.1] respectively.

Theorem 2.1. *For any $\underline{\rho} \in \widehat{\mathbb{P}}_n^\sigma$,*

$$\langle \chi_{\underline{\nu}}^{\underline{\rho}}, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle = \begin{cases} 1, & \text{if all } \rho_\xi \text{ are even and } \prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{|\rho_\xi|}{2}} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

(Note that if all ρ_ξ are even, $\prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{|\rho_\xi|}{2}}$ is either 1 or η , since its square is $\Pi(\underline{\rho}) = 1$.)

Theorem 2.2. *For any $\underline{\rho} \in \widehat{\mathbb{P}}_n^\sigma$ and $\epsilon \in \{\pm 1\}$,*

$$\begin{aligned} \langle \chi_{\underline{\nu}}^{\underline{\rho}}, \text{Ind}_{PGL_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle = & \begin{cases} \frac{1}{4} \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1}} \left(\prod_i (m_i(\rho_\xi) + 1) \right), & \text{if } d_\xi = -1 \Rightarrow \rho'_\xi \text{ is even} \\ 0, & \text{otherwise} \end{cases} \\ & + \begin{cases} \frac{1}{2}\epsilon, & \text{if all } \rho'_\xi \text{ are even and } \prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{|\rho_\xi|}{2}} = 1 \\ 0, & \text{otherwise} \end{cases} \\ & + \begin{cases} \frac{\pm 1}{4} \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1 \\ 2 \nmid m_\xi}} \left(\prod_i (m_{2i}(\rho_\xi) + 1) \right) \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1 \\ 2 \mid m_\xi}} \left(\prod_i (m_i(\rho_\xi) + 1) \right) \\ \text{if } d_\xi = 1, 2 \nmid m_\xi \Rightarrow 2 \mid m_{2i+1}(\rho_\xi), \forall i, \\ \text{and } d_\xi = -1 \Rightarrow \rho'_\xi \text{ even} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where the sign ± 1 in the third term is

$$(-1)^{\frac{n}{2}} \Phi(\underline{\rho}) \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1 \\ 2 \nmid m_\xi}} (-1)^{\ell(\rho_\xi)_{2 \bmod 4}}.$$

(Note that the third term can only be nonzero if all $m_\xi |\rho_\xi|$ are even, so $\Phi(\underline{\rho})$ is defined.)

Example. The elements of $\widehat{\mathbb{P}}_2^\sigma$ can be written in ‘exponential notation’ as follows:

$$1^{(2)}, 1^{(1^2)}, \eta^{(2)}, \eta^{(1^2)}, \text{ and } \xi^{(1)}(\xi^{-1})^{(1)} \text{ for } \xi \in (L^\sigma \cup L^{\tilde{\sigma}}) \setminus \{1, \eta\}.$$

Here $L^{\tilde{\sigma}} = \{\xi \in L \mid \xi^q = \xi^{-1}\}$. Theorem 2.1 states that the only irreducible constituent of $\text{Ind}_{PGL_2(\mathbb{F}_q)}^{PGL_2(\mathbb{F}_q)}(1)$ is the trivial character $\chi^{1^{(2)}}$, which is correct because

$PGSp_2(\mathbb{F}_q) = PGL_2(\mathbb{F}_q)$. In the following table, each row contains, for a particular element of $\widehat{\mathbb{P}}_2^\sigma$, the value of the three terms of the right-hand side of Theorem 2.2, and their sum:

$1^{(2)}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1
$1^{(1^2)}$	$\frac{3}{4}$	$\frac{1}{2}\epsilon$	$-\frac{1}{4}$	$\begin{matrix} 1, & \text{if } \epsilon = 1 \\ 0, & \text{if } \epsilon = -1 \end{matrix}$
$\eta^{(2)}$	$\begin{cases} \frac{1}{2}, & \text{if } d_\eta = 1 \\ 0, & \text{if } d_\eta = -1 \end{cases}$	0	$\begin{cases} -\frac{1}{2} \\ 0 \end{cases}$	0
$\eta^{(1^2)}$	$\begin{cases} \frac{3}{4}, & \text{if } d_\eta = 1 \\ \frac{1}{4}, & \text{if } d_\eta = -1 \end{cases}$	0	$\begin{cases} \frac{1}{4} \\ -\frac{1}{4} \end{cases}$	$\begin{matrix} 1, & \text{if } d_\eta = 1 \\ 0, & \text{if } d_\eta = -1 \end{matrix}$
$\begin{matrix} \xi^{(1)}(\xi^{-1})^{(1)}, \\ \xi \in L^\sigma \setminus \{1, \eta\} \end{matrix}$	$\begin{cases} 1, & \text{if } d_\xi = 1 \\ 0, & \text{if } d_\xi = -1 \end{cases}$	0	0	$\begin{matrix} 1, & \text{if } d_\xi = 1 \\ 0, & \text{if } d_\xi = -1 \end{matrix}$
$\begin{matrix} \xi^{(1)}(\xi^{-1})^{(1)}, \\ \xi \in L^{\tilde{\sigma}} \setminus \{1, \eta\} \end{matrix}$	$\frac{1}{2}$	0	$-\frac{1}{2}\tilde{d}_\xi$	$\begin{matrix} 0, & \text{if } \tilde{d}_\xi = 1 \\ 1, & \text{if } \tilde{d}_\xi = -1 \end{matrix}$

In the third and fourth rows we have used the fact that $\langle \sqrt{\beta}, \eta \rangle_2 = \langle -\beta, \eta \rangle = -d_\eta$, for $\sqrt{\beta}$ as above. In the last row, we have used the facts that $d_\xi = 1$ and $\langle \sqrt{\beta}, \xi \rangle_2 = \tilde{d}_\xi$, where \tilde{d}_ξ is the sign in the equation $\sqrt{\xi}^q = \pm \sqrt{\xi}^{-1}$. So Theorem 2.2 implies:

$$\begin{aligned}
 \text{Ind}_{PGO_2^+(\mathbb{F}_q)}^{PGL_2(\mathbb{F}_q)}(1) &= \chi^{1^{(2)}} + \chi^{1^{(1^2)}} + \begin{cases} \chi^{\eta^{(1^2)}}, & \text{if } d_\eta = 1 \\ 0, & \text{if } d_\eta = -1 \end{cases} \\
 &\quad + \sum_{\substack{\xi \in L^\sigma \setminus \{1, \eta\} \\ d_\xi = 1}} \chi^{\xi^{(1)}(\xi^{-1})^{(1)}} + \sum_{\substack{\xi \in L^{\tilde{\sigma}} \setminus \{1, \eta\} \\ \tilde{d}_\xi = -1}} \chi^{\xi^{(1)}(\xi^{-1})^{(1)}}, \\
 (2.7) \quad \text{Ind}_{PGO_2^-(\mathbb{F}_q)}^{PGL_2(\mathbb{F}_q)}(1) &= \chi^{1^{(2)}} + \begin{cases} \chi^{\eta^{(1^2)}}, & \text{if } d_\eta = 1 \\ 0, & \text{if } d_\eta = -1 \end{cases} \\
 &\quad + \sum_{\substack{\xi \in L^\sigma \setminus \{1, \eta\} \\ d_\xi = 1}} \chi^{\xi^{(1)}(\xi^{-1})^{(1)}} + \sum_{\substack{\xi \in L^{\tilde{\sigma}} \setminus \{1, \eta\} \\ \tilde{d}_\xi = -1}} \chi^{\xi^{(1)}(\xi^{-1})^{(1)}}.
 \end{aligned}$$

In the sums, only one representative of each pair $\{\xi, \xi^{-1}\}$ should be taken. Of course, these decompositions are known and easy to calculate directly. (As Theorem 2.2 makes clear, the fact that they are multiplicity-free is specific to the case $n = 2$.)

As mentioned in the introduction, we will prove Theorems 2.1 and 2.2 in the equivalent forms involving the basic characters $B_{\underline{\nu}}$. Applying (2.6), these are:

Theorem 2.3. *For any $\underline{\nu} \in \widehat{\mathbb{P}}_n^\sigma$,*

$$\langle B_{\underline{\nu}}, \text{Ind}_{PGSp_n(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle = \begin{cases} \prod_{\xi \in \langle \sigma \rangle \setminus L} \sum_{\substack{\rho_\xi \vdash |\nu_\xi| \\ \rho_\xi \text{ even}}} \chi_{\nu_\xi}^{\rho_\xi}, & \text{if all } |\nu_\xi| \text{ are even and} \\ & \prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{|\nu_\xi|}{2}} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.4. *For any $\underline{\nu} \in \widehat{\mathbb{P}}_n^\sigma$ and $\epsilon \in \{\pm 1\}$,*

$$\begin{aligned}
\langle B_{\underline{\nu}}, \text{Ind}_{PGO_n^\epsilon(\mathbb{F}_q)}^{PGL_n(\mathbb{F}_q)}(1) \rangle &= \frac{1}{4} \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1}} (-1)^{|\nu_\xi|} \sum_{\rho_\xi \vdash |\nu_\xi|} \left(\prod_i (m_i(\rho_\xi) + 1) \right) \chi_{\nu_\xi}^{\rho_\xi} \\
&\quad \times \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = -1}} \sum_{\substack{\rho'_\xi \vdash |\nu_\xi| \\ \rho'_\xi \text{ even}}} \chi_{\nu_\xi}^{\rho'_\xi} \\
&\quad + \begin{cases} \frac{1}{2}^\epsilon \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ \rho'_\xi \vdash |\nu_\xi| \\ \rho'_\xi \text{ even}}} \chi_{\nu_\xi}^{\rho'_\xi}, & \text{if all } |\nu_\xi| \text{ are even and} \\ & \prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{|\nu_\xi|}{2}} = 1, \\ 0, & \text{otherwise} \end{cases} \\
&\quad + \frac{\Phi(\underline{\nu})}{4} \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1 \\ 2 \nmid m_\xi}} \sum_{\substack{\rho_\xi \vdash |\nu_\xi| \\ 2 \mid m_{2i+1}(\rho_\xi), \forall i}} (-1)^{\frac{1}{2}|\rho_\xi| + \ell(\rho_\xi) \bmod 4} \left(\prod_i (m_{2i}(\rho_\xi) + 1) \right) \chi_{\nu_\xi}^{\rho_\xi} \\
&\quad \times \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1 \\ 2 \mid m_\xi}} (-1)^{|\nu_\xi| + \frac{1}{2}m_\xi|\nu_\xi|} \sum_{\rho_\xi \vdash |\nu_\xi|} \left(\prod_i (m_i(\rho_\xi) + 1) \right) \chi_{\nu_\xi}^{\rho_\xi} \\
&\quad \times \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = -1}} (-1)^{\frac{1}{2}m_\xi|\nu_\xi|} \sum_{\substack{\rho_\xi \vdash |\nu_\xi| \\ \rho_\xi \text{ even}}} \chi_{\nu_\xi}^{\rho_\xi}.
\end{aligned}$$

Here the sign $(-1)^{n + \sum_{\xi \in \langle \sigma \rangle \setminus L} |\nu_\xi|}$ of (2.6) has been rewritten as $\prod_{\xi \in \langle \sigma \rangle \setminus L} (-1)^{|\nu_\xi|}$ and then distributed among the factors as appropriate. Also, in the last term of Theorem 2.4, the sign $(-1)^{\frac{n}{2}}$ of Theorem 2.2 has been rewritten as $\prod_{\xi \in \langle \sigma \rangle \setminus L} (-1)^{\frac{1}{2}m_\xi|\nu_\xi|}$ and then distributed among the factors (each of which can only be nonzero when $m_\xi|\nu_\xi|$ is even, so this and $\Phi(\underline{\nu})$ make sense). Theorems 2.3 and 2.4 will be deduced from Lusztig's formula in the next two sections.

3. PROOF OF THEOREM 2.3

The proof of Theorem 2.3 is mostly identical to that of [2, (2.1.1)], but we will go through all the steps again as a warm-up for the more complicated arguments in the next section. We fix a non-degenerate **skew-symmetric** bilinear form B on V , compatible with F in the sense that $B(F(v), F(v')) = B(v, v')^q$. Let $\theta : G \rightarrow G$ be the involution defined by

$$B(\theta(g)v, v') = B(v, g^{-1}v'), \quad \forall g \in G, v, v' \in V,$$

and let $K = G^\theta = Sp(V, B)$. We have $F\theta = \theta F$, so K is F -stable, and K^F is what we have been calling $Sp_n(\mathbb{F}_q)$.

Now for $z \in Z$, $\theta(z) = z^{-1}$, so $K \cap Z = \{\pm 1\}$. If we write θ also for the induced involution of \overline{G} , it is clear that $\overline{G}^\theta = \overline{K} = KZ/Z$, since

$$KZ = \{g \in G \mid B(gv, gv') = \alpha B(v, v'), \quad \forall v, v' \in V, \exists \alpha \in k^\times\},$$

the group called either GSp_n or CSp_n . However, as mentioned in the introduction,

$$K^F Z^F = \{g \in G^F \mid B(gv, gv') = \alpha B(v, v'), \quad \forall v, v' \in V, \exists \alpha \in (\mathbb{F}_q^\times)^2\}$$

is a subgroup of index 2 in

$$(KZ)^F = \{g \in G^F \mid B(gv, gv') = \alpha B(v, v'), \forall v, v' \in V, \exists \alpha \in \mathbb{F}_q^\times\},$$

so the image of $K^F \rightarrow \overline{K}^F$ is of index 2 in \overline{K}^F . The group \overline{K}^F is what we have been calling $PGSp_n(\mathbb{F}_q)$.

Now let $\underline{\nu} \in \widehat{\mathbb{P}}_n^\sigma$, and fix an F -stable maximal torus \overline{T} of \overline{G} and a character $\lambda \in \widehat{\overline{T}^F}$ such that the pair (\overline{T}, λ) is in the \overline{G}^F -orbit corresponding to $\underline{\nu}$ under the bijection described in the previous section. Let T be the preimage of \overline{T} in G , a maximal torus of G . We want to deduce Theorem 2.3 from Lusztig's formula [4, Theorem 3.3] for the inner product $\langle B_{\underline{\nu}}, \text{Ind}_{\overline{K}^F}^{\overline{G}^F}(1) \rangle$. A key ingredient in the formula is the set

$$\Theta_{\overline{T}} := \{\bar{f} \in \overline{G} \mid \theta(\bar{f}^{-1}\overline{T}\bar{f}) = \bar{f}^{-1}\overline{T}\bar{f}\}.$$

Since $T \supset Z$, it is obvious that $\Theta_{\overline{T}}$ is the image of the analogous set

$$\Theta_T := \{f \in G \mid \theta(f^{-1}Tf) = f^{-1}Tf\}.$$

For $f \in \Theta_T$, $f^{-1}Tf$ is a θ -stable maximal torus of G , or in other words the stabilizer of a decomposition $V = \bigoplus_{i=1}^n L_i$ into lines with the property that $L_j^\perp = \bigoplus_{i \neq w(j)} L_i$ for some involution $w \in S_n$ with no fixed points. (Here L_j^\perp is the subspace perpendicular to L_j under B .) If $t \in f^{-1}Tf$ has eigenvalue α_i on L_i for all i , then $\theta(t)$ has eigenvalue $\alpha_{w(i)}^{-1}$ on L_i for all i ; thus t lies in $f^{-1}Tf \cap K$ if and only if $\alpha_i \alpha_{w(i)} = 1$ for all i , and the image \bar{t} of t lies in $\bar{f}^{-1}\overline{T}\bar{f} \cap \overline{K}$ if and only if $\alpha_i \alpha_{w(i)} = \beta$ for all i , for some $\beta \in k^\times$. From this the following result is obvious.

Proposition 3.1. *For all $\bar{f} \in \Theta_{\overline{T}}$, $\bar{f}^{-1}\overline{T}\bar{f} \cap \overline{K}$ is connected and $Z_{\overline{G}}(\bar{f}^{-1}\overline{T}\bar{f} \cap \overline{K}) = \bar{f}^{-1}\overline{T}\bar{f}$.*

Under these circumstances, Lusztig's formula takes a simpler form. Define

$$\Theta_{\overline{T}, \lambda}^F := \{\bar{f} \in \Theta_{\overline{T}}^F \mid \lambda|_{(\overline{T} \cap \bar{f}^{-1}\overline{K}\bar{f})^F} = 1\}.$$

This is clearly a union of $\overline{T}^F - \overline{K}^F$ double cosets, and [4, Theorem 3.3] implies that

$$(3.1) \quad \langle B_{\underline{\nu}}, \text{Ind}_{\overline{K}^F}^{\overline{G}^F}(1) \rangle = |\overline{T}^F \backslash \Theta_{\overline{T}, \lambda}^F / \overline{K}^F|.$$

We now aim to express the right-hand side in terms of $\underline{\nu}$.

Let $W(T) = N_G(T)/T$ be the Weyl group of T (isomorphic to the symmetric group S_n), and let $W(T)_{\text{ff-inv}}$ be the subset corresponding to the fixed-point-free involutions in S_n . For any $f \in \Theta_T$, we define $w_f \in W(T)_{\text{ff-inv}}$ by requiring that $f^{-1}w_f f \in W(f^{-1}Tf)$ corresponds to the above involution w of the fixed lines of $f^{-1}Tf$. It is clear that w_f depends only on $\bar{f} \in \Theta_{\overline{T}}$.

Proposition 3.2. *The map $\bar{f} \mapsto w_f$ induces a bijection $\overline{T} \backslash \Theta_{\overline{T}} / \overline{K} \xrightarrow{\sim} W(T)_{\text{ff-inv}}$.*

Proof. This follows trivially from the corresponding statement for G , which is [2, Proposition 2.0.1]. \square

Proposition 3.3. *The map $\bar{f} \mapsto w_f$ induces a bijection $\overline{T}^F \backslash \Theta_{\overline{T}}^F / \overline{K}^F \xrightarrow{\sim} W(T)_{\text{ff-inv}}^F$.*

Proof. Clearly the map in Proposition 3.2 is F -stable, so we get a bijection

$$(\overline{T} \backslash \Theta_{\overline{T}} / \overline{K})^F \xrightarrow{\sim} W(T)_{\text{ff-inv}}^F.$$

Now regard $\overline{T} \backslash \Theta_{\overline{T}} / \overline{K}$ as the set of orbits of the group $\overline{T} \times \overline{K}$ on $\Theta_{\overline{T}}$. Since $\overline{T} \times \overline{K}$ is connected, Lang's Theorem implies that $\overline{T}^F \backslash \Theta_{\overline{T}}^F / \overline{K}^F \rightarrow (\overline{T} \backslash \Theta_{\overline{T}} / \overline{K})^F$ is surjective. To prove injectivity, we need only check that the stabilizers in $\overline{T} \times \overline{K}$ of points in $\Theta_{\overline{T}}$ are connected, which is exactly what the first part of Proposition 3.1 says. (Note that in the proof of the corresponding result [2, Lemma 2.1.2], the words 'injective' and 'surjective' need to be swapped!) \square

The next step is to identify the subset of $W(T)_{\text{ff-inv}}^F$ which corresponds under this bijection to the set we are trying to enumerate, $\overline{T}^F \backslash \Theta_{\overline{T}, \lambda}^F / \overline{K}^F$. Define

$$W(T)_{\lambda, \text{ff-inv}}^F := \{w \in W(T)_{\text{ff-inv}}^F \mid \lambda(wt) = \lambda(t), \forall t \in T^F\}.$$

(Here we are abusing notation by writing λ also for the pull-back of λ to T^F .) Now [2, Lemma 2.1.3] states that for $f \in \Theta_T^F$, $f \in \Theta_{T, \lambda}^F \iff w_f \in W(T)_{\lambda, \text{ff-inv}}^F$. This follows immediately from the fact that

$$(3.2) \quad (T \cap fKf^{-1})^F = \{t \in T^F \mid w_f t = t^{-1}\} = \{(w_f t)t^{-1} \mid t \in T^F\}.$$

However, for $\bar{f} \in \Theta_{\overline{T}}^F$ the analogous result is not quite true, since

$$\overline{T}_{\bar{f}}^F := \{(w_f \bar{t})\bar{t}^{-1} \mid \bar{t} \in \overline{T}^F\}$$

is a subgroup of index 2 in

$$(\overline{T} \cap \bar{f}\overline{K}\bar{f}^{-1})^F = \{\bar{t} \in \overline{T}^F \mid w_f \bar{t} = \bar{t}^{-1}\}.$$

To see this, recall that the fixed lines of T can be labelled $L_{(\xi, j, i)}$ as in (2.2), so $W(T)$ can be viewed as the group of permutations of the triples (ξ, j, i) . If $t \in T^F$ has eigenvalue $\alpha_{(\xi, j, i)}$ on $L_{(\xi, j, i)}$, then

$$(3.3) \quad w_f \bar{t} = \bar{t}^{-1} \iff \alpha_{(\xi, j, i)} \alpha_{w_f(\xi, j, i)} = \beta, \forall (\xi, j, i), \text{ for some } \beta \in \mathbb{F}_q^\times,$$

whereas

$$(3.4) \quad \bar{t} \in \overline{T}_{\bar{f}}^F \iff \alpha_{(\xi, j, i)} \alpha_{w_f(\xi, j, i)} = \beta, \forall (\xi, j, i), \text{ for some } \beta \in (\mathbb{F}_q^\times)^2.$$

Consequently, the correct analogue of [2, Lemma 2.1.3] is:

Lemma 3.4. *For $\bar{f} \in \Theta_{\overline{T}}^F$, $\bar{f} \in \Theta_{\overline{T}, \lambda}^F$ if and only if $w_f \in W(T)_{\lambda, \text{ff-inv}}^F$ **and** $\lambda(t_0) = 1$ for some (hence any) $t_0 \in T^F$ whose eigenvalues $\{\alpha_{(\xi, j, i)}\}$ have the property that $\alpha_{(\xi, j, i)} \alpha_{w_f(\xi, j, i)} = \beta$, $\forall (\xi, j, i)$, for some $\beta \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$.*

As in [2, §1.4], the set $W(T)_{\lambda, \text{ff-inv}}^F$ can be described using (2.2)–(2.4). Let Λ be the set of pairs (ξ, j) where $\xi \in \langle \sigma \rangle \backslash L$ and $1 \leq j \leq \ell(\nu_\xi)$. (In [2], this and the related sets were denoted $\Lambda(\underline{\nu})$, etc.) Any $w \in W(T)_{\lambda, \text{ff-inv}}^F$ defines a partition $\Lambda = \Lambda_w^2 \amalg \Lambda_w^3$ such that:

- for all $(\xi, j) \in \Lambda_w^2$, $(\nu_\xi)_j$ is even and

$$w(\xi, j, i) = (\xi, j, i + \frac{1}{2}m_\xi(\nu_\xi)_j) \text{ for all } i \in \mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z};$$

- for all $(\xi, j) \in \Lambda_w^3$,

$$w(\xi, j, i) = (\xi, \tilde{j}, i + i(w, \xi, j)) \text{ for all } i \in \mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z},$$

where $(\xi, \tilde{j}) \in \Lambda_w^3$, $\tilde{j} \neq j$, $(\nu_\xi)_{\tilde{j}} = (\nu_\xi)_j$, and $i(w, \xi, j)$ is some fixed ‘shift’ in $m_\xi\mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z}$.

(The set Λ_w^1 will appear in the next section; it is empty here because we are dealing with fixed-point-free involutions.) Clearly $(\xi, j) \mapsto (\xi, \tilde{j})$ is a fixed-point-free involution of Λ_w^3 ; we will write $\widetilde{\Lambda_w^3}$ for an arbitrarily chosen set of representatives of its orbits. Note that the existence of $w \in W(T)_{\lambda, \text{ff-inv}}^F$ forces each $|\nu_\xi|$ to be even.

As explained in [2, §1.4, §2.1], the above description gives a bijection between $W(T)_{\lambda, \text{ff-inv}}^F$ and $\prod_{\xi \in \langle \sigma \rangle \setminus L} Z_{\text{ff-inv}}^{\nu_\xi}$, where $Z_{\text{ff-inv}}^{\nu}$ denotes the set of fixed-point-free involutions in $S_{|\nu|}$ which commute with a given element w_ν of cycle-type ν ; essentially this bijection is defined by dividing the shifts $\frac{1}{2}m_\xi(\nu_\xi)_j$ and $i(w, \xi, j)$ by m_ξ .

Now suppose t_0 , $\alpha_{(\xi, j, i)}$, β are as in Lemma 3.4. Since $t_0 \in T^F$, we have $\alpha_{(\xi, j, i+1)} = \alpha_{(\xi, j, i)}^q$ for all (ξ, j, i) . Clearly the condition $\alpha_{(\xi, j, i)} \alpha_{w_f(\xi, j, i)} = \beta$ is equivalent to the conjunction of the following conditions:

- for all $(\xi, j) \in \Lambda_{w_f}^2$, $\alpha_{(\xi, j, 1)}^{1+q^{\frac{1}{2}m_\xi(\nu_\xi)_j}} = \beta$;
- for all $(\xi, j) \in \Lambda_{w_f}^3$, $\alpha_{(\xi, j, 1)} \alpha_{(\xi, \tilde{j}, 1)}^{q^{i(w_f, \xi, j)}} = \beta$.

Hence

$$\begin{aligned} \lambda(t_0) &= \prod_{\xi \in \langle \sigma \rangle \setminus L} \prod_{j=1}^{\ell(\nu_\xi)} \langle \alpha_{(\xi, j, 1)}, \xi \rangle_{m_\xi(\nu_\xi)_j} \\ &= \prod_{(\xi, j) \in \Lambda_{w_f}^2} \langle \alpha_{(\xi, j, 1)}^{1+q^{m_\xi} + q^{2m_\xi} + \dots + q^{((\nu_\xi)_j - 1)m_\xi}}, \xi \rangle_{m_\xi} \\ &\quad \times \prod_{(\xi, j) \in \widetilde{\Lambda_{w_f}^3}} \langle \alpha_{(\xi, j, 1)}^{1+q^{m_\xi} + q^{2m_\xi} + \dots + q^{((\nu_\xi)_j - 1)m_\xi}} \alpha_{(\xi, \tilde{j}, 1)}^{1+q^{m_\xi} + q^{2m_\xi} + \dots + q^{((\nu_\xi)_{\tilde{j}} - 1)m_\xi}}, \xi \rangle_{m_\xi} \\ &= \prod_{(\xi, j) \in \Lambda_{w_f}^2} \langle \beta^{\frac{1}{2}(\nu_\xi)_j}, \xi \rangle_{m_\xi} \prod_{(\xi, j) \in \widetilde{\Lambda_{w_f}^3}} \langle \beta^{(\nu_\xi)_j}, \xi \rangle_{m_\xi} \\ &= \prod_{(\xi, j) \in \Lambda_{w_f}^2} \langle \beta, N(\xi) \rangle^{\frac{1}{2}(\nu_\xi)_j} \prod_{(\xi, j) \in \widetilde{\Lambda_{w_f}^3}} \langle \beta, N(\xi) \rangle^{(\nu_\xi)_j} \\ &= \langle \beta, \prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{1}{2}|\nu_\xi|} \rangle. \end{aligned}$$

Note that $\prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{1}{2}|\nu_\xi|}$ is a square root of $\Pi(\underline{\nu}) = 1$. So Lemma 3.4 can be restated as follows:

Proposition 3.5. *If all $|\nu_\xi|$ are even and $\prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{1}{2}|\nu_\xi|} = 1$, $\overline{T}^F \setminus \Theta_{T, \lambda}^F / \overline{K}^F$ is in bijection with $W(T)_{\lambda, \text{ff-inv}}^F$. Otherwise, $\Theta_{T, \lambda}^F$ is empty.*

Combining this with Lusztig’s formula (3.1), we deduce

Proposition 3.6. *For any $\underline{\nu} \in \widehat{\mathbb{P}}_n^\sigma$,*

$$\langle B_{\underline{\nu}}, \text{Ind}_{\overline{K}^F}^{\overline{G}^F}(1) \rangle = \begin{cases} \prod_{\xi \in \langle \sigma \rangle \setminus L} |Z_{\text{ff-inv}}^{\nu_\xi}|, & \text{if all } |\nu_\xi| \text{ are even and} \\ 0, & \text{otherwise.} \end{cases} \quad \prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{1}{2}|\nu_\xi|} = 1,$$

This implies Theorem 2.3 by the same combinatorial fact used in [2, §2.1]: namely,

$$(3.5) \quad |Z_{\text{ff-inv}}^\nu| = \sum_{\substack{\rho \vdash |\nu| \\ \rho \text{ even}}} \chi_\nu^\rho.$$

A reference for this fact is [5, VII(2.4)].

4. PROOF OF THEOREM 2.4

In this section we fix a non-degenerate **symmetric** bilinear form B on V , compatible with F in the sense that $B(F(v), F(v')) = B(v, v')^q$. Let $\theta : G \rightarrow G$ be the involution defined by

$$B(\theta(g)v, v') = B(v, g^{-1}v'), \quad \forall g \in G, v, v' \in V,$$

and let $K = G^\theta = O(V, B)$. Note that K has two components, with $K^\circ = SO(V, B)$; since $K \cap Z = \{\pm 1\} \subset K^\circ$, $\overline{G}^\theta = \overline{K} = KZ/Z$ has two components also. This group \overline{K} is what is known as PGO_n .

We have $F\theta = \theta F$, so K is F -stable. The Witt index of B on V^F is either $n/2$ or $n/2 - 1$, and accordingly $\overline{K}^F \cong PGO_n^\epsilon(\mathbb{F}_q)$ where $\epsilon = +1$ or -1 . As in the previous section, the image of $K^F \rightarrow \overline{K}^F$ is an index-2 subgroup of \overline{K}^F (**not** equal to the index-2 subgroup $(\overline{K}^\circ)^F$).

Fix $\underline{\nu} \in \widehat{\mathbb{P}}_n^\sigma$, and define $\overline{T}, \lambda, T, \Theta_{\overline{T}}, \Theta_T, W(T) \cong S_n$ as in the previous section. Let $W(T)_{\text{inv}}$ denote the set of involutions in $W(T)$ (including 1). For any $f \in \Theta_T$, $f^{-1}Tf$ is the stabilizer of a decomposition $V = \bigoplus_{i=1}^n L_i$ into lines with the property that $L_j^\perp = \bigoplus_{i \neq w(j)} L_i$ for some involution $w \in S_n$, and we define $w_f \in W(T)_{\text{inv}}$ by requiring that $f^{-1}w_f f \in W(f^{-1}Tf)$ corresponds to w . It is clear that w_f depends only on $\bar{f} \in \Theta_{\overline{T}}$.

Proposition 4.1. *The map $\bar{f} \mapsto w_f$ induces a bijection $\overline{T} \setminus \Theta_{\overline{T}} / \overline{K} \xrightarrow{\sim} W(T)_{\text{inv}}$.*

Proof. This follows trivially from the corresponding statement for G , which is [2, Proposition 4.0.2]. \square

However, the analogue of Proposition 3.3 is false, since neither \overline{K} nor $\overline{T} \cap \bar{f}\overline{K}\bar{f}^{-1}$ for general $\bar{f} \in \Theta_{\overline{T}}^F$ is connected. Insead, we have the following result, where $\epsilon_{\underline{\nu}}$ denotes the sign of the permutation by which F acts on the fixed lines of T :

Proposition 4.2. *The map $\bar{f} \mapsto w_f$ induces a map $\overline{T}^F \setminus \Theta_{\overline{T}}^F / \overline{K}^F \rightarrow W(T)_{\text{inv}}^F$. If $\epsilon_{\underline{\nu}} = \epsilon$, this map is surjective. If $\epsilon_{\underline{\nu}} = -\epsilon$, the image is $W(T)_{\text{inv}}^F \setminus W(T)_{\text{ff-inv}}^F$.*

Proof. This follows trivially from the corresponding statement for G^F , which is [2, Lemma 4.2.2]. \square

To state Lusztig's formula in this case, we need to introduce, for any $\bar{f} \in \Theta_{\bar{T}}^F$, the function $\epsilon_{\bar{T}, \bar{f}} : (\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^F \rightarrow \{\pm 1\}$ defined by

$$\epsilon_{\bar{T}, \bar{f}}(\bar{t}) = (-1)^{\mathbb{F}_q\text{-rank}(Z_{\bar{G}}((\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ)) + \mathbb{F}_q\text{-rank}(Z_{\bar{G}}^\circ(\bar{t}) \cap Z_{\bar{G}}((\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ))}.$$

It follows from [4, Proposition 2.3] that $\epsilon_{\bar{T}, \bar{f}}$ is a group homomorphism which factors through $(\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^F / ((\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ)^F$. We can then define

$$\Theta_{\bar{T}, \lambda}^F := \{\bar{f} \in \Theta_{\bar{T}}^F \mid \lambda|_{(\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^F} = \epsilon_{\bar{T}, \bar{f}}\}.$$

This is clearly a union of $\bar{T}^F - \bar{K}^F$ double cosets, and [4, Theorem 3.3] says that

$$(4.1) \quad \langle B_{\bar{T}}, \text{Ind}_{\bar{K}^F}^{\bar{G}^F}(1) \rangle = \sum_{\bar{f} \in \bar{T}^F \setminus \Theta_{\bar{T}, \lambda}^F / \bar{K}^F} (-1)^{\mathbb{F}_q\text{-rank}(\bar{T}) + \mathbb{F}_q\text{-rank}(Z_{\bar{G}}((\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ))}.$$

Our immediate aim is to use the map $\bar{f} \mapsto w_f$ to turn the right-hand side into a sum over a suitable subset of $W(T)_{\text{inv}}^F$.

As in the previous section, we view $W(T)$ as the group of permutations of the triples (ξ, j, i) in (2.2). If $\bar{f} \in \Theta_{\bar{T}}^F$, and $t \in T$ has eigenvalue $\alpha_{(\xi, j, i)}$ on $L_{(\xi, j, i)}$, then

$$(4.2) \quad \bar{t} \in \bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1} \iff \alpha_{(\xi, j, i)} \alpha_{w_f(\xi, j, i)} = \beta, \forall (\xi, j, i), \text{ for some } \beta \in k^\times.$$

Unlike in the previous section, w_f is now allowed to have fixed points; the eigenvalues $\alpha_{(\xi, j, i)}$ where $w_f(\xi, j, i) = (\xi, j, i)$ must all be square roots of β . It is clear that the additional condition required in order that $\bar{t} \in (\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ$ is that all these square roots are equal.

This allows us to prove a partial analogue of Lemma 3.4. Define

$$W(T)_{\lambda, \text{inv}}^F := \{w \in W(T)_{\text{inv}}^F \mid \lambda(wt) = \lambda(t), \forall t \in T^F\}.$$

(Once again we are blurring the distinction between λ and its pull-back to T^F .)

Lemma 4.3. *If $\bar{f} \in \Theta_{\bar{T}, \lambda}^F$, then $w_f \in W(T)_{\lambda, \text{inv}}^F$.*

Proof. From our description of $(\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ$, it is clear that for all $t \in T$ the element $(w_f t)\bar{t}^{-1}$ lies in $(\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ$. Since $\epsilon_{\bar{T}, \bar{f}}$ is trivial on $((\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ)^F$, the assumption $\bar{f} \in \Theta_{\bar{T}, \lambda}^F$ implies that $\lambda((w_f t)\bar{t}^{-1}) = 1$ for all $t \in T^F$. Thus $w_f \in W(T)_{\lambda, \text{inv}}^F$. \square

The set $W(T)_{\lambda, \text{inv}}^F$ contains the set $W(T)_{\lambda, \text{ff-inv}}^F$ used in the previous section and can be described similarly. Namely, any $w \in W(T)_{\lambda, \text{inv}}^F$ defines a partition $\Lambda = \Lambda_w^1 \amalg \Lambda_w^2 \amalg \Lambda_w^3$ such that for $(\xi, j) \in \Lambda_w^1$, $w(\xi, j, i) = (\xi, j, i)$ for all $i \in \mathbb{Z}/m_\xi(\nu_\xi)_j\mathbb{Z}$, and Λ_w^2 and Λ_w^3 are as before. Define $\widetilde{\Lambda_w^3}$ as before also, and let $\ell_w^i = |\Lambda_w^i|$. Note that since n is even,

$$(4.3) \quad \sum_{(\xi, j) \in \Lambda_w^1} m_\xi(\nu_\xi)_j \text{ is even.}$$

As explained in [2, §1.4], this description gives a bijection between $W(T)_{\lambda, \text{inv}}^F$ and $\prod_{\xi \in \langle \sigma \rangle \setminus L} Z_{\text{inv}}^{\nu_\xi}$, where Z_{inv}^{ν} denotes the set of fixed-point-free involutions in $S_{|\nu|}$ which commute with a given element w_ν of cycle-type ν ; this bijection is defined by dividing all shifts by m_ξ .

Now we define a subset X of $W(T)_{\lambda, \text{inv}}^F$ by the rule that $w \in X$ if and only if $(\nu_\xi)_j$ is even for all $(\xi, j) \in \Lambda_w^1$ with $d_\xi = -1$. (In [2, §4] this was called X_{inv}^μ .)

Also let Y be the subset of $W(T)_{\lambda, \text{inv}}^F$ defined by the requirement that $m_\xi(\nu_\xi)_j$ is even for all $(\xi, j) \in \Lambda_w^1$. Trivially $W(T)_{\lambda, \text{ff-inv}}^F \subseteq X \cap Y$. We can measure the non-injectivity of the map in Proposition 4.2:

Proposition 4.4. *Let $w \in W(T)_{\lambda, \text{inv}}^F$. The number of double cosets $\overline{T}^F \bar{f} \overline{K}^F$ in $\overline{T}^F \backslash \Theta_{\overline{T}}^F / \overline{K}^F$ such that $w_f = w$ equals*

$$\begin{cases} 0, & \text{if } w \in W(T)_{\lambda, \text{ff-inv}}^F \text{ and } \epsilon_{\underline{\nu}} = -\epsilon, \\ 1, & \text{if } w \in W(T)_{\lambda, \text{ff-inv}}^F \text{ and } \epsilon_{\underline{\nu}} = \epsilon, \\ 2^{\ell_w^1 - 1}, & \text{if } w \in Y \setminus W(T)_{\lambda, \text{ff-inv}}^F, \\ 2^{\ell_w^1 - 2}, & \text{if } w \in W(T)_{\lambda, \text{inv}}^F \setminus Y. \end{cases}$$

Proof. Let $\bar{f} \in \Theta_{\overline{T}}^F$ be such that $w_f \in W(T)_{\lambda, \text{inv}}^F$. The number of $\overline{T}^F - \overline{K}^F$ double cosets in $(\overline{T} \bar{f} \overline{K})^F$ can be calculated by the same method as in [2, Lemma 4.1.3]: it is the index in the group

$$H := \left\{ (\epsilon_{(\xi, j, i)}) \in \prod_{(\xi, j) \in \Lambda_{w_f}^1} (\pm 1)^{m_\xi(\nu_\xi)_j} \mid \prod_{(\xi, j, i)} \epsilon_{(\xi, j, i)} = 1 \right\}$$

of the subgroup H' defined by the further requirement that either $\prod_i \epsilon_{(\xi, j, i)} = 1$ for all $(\xi, j) \in \Lambda_{w_f}^1$ or $\prod_i \epsilon_{(\xi, j, i)} = (-1)^{m_\xi(\nu_\xi)_j}$ for all $(\xi, j) \in \Lambda_{w_f}^1$. If $w_f \in W(T)_{\lambda, \text{ff-inv}}^F$, then $\Lambda_{w_f}^1$ is empty and H is trivial, so the index is 1. Otherwise,

$$(4.4) \quad |H| = 2^{\sum_{(\xi, j) \in \Lambda_{w_f}^1} m_\xi(\nu_\xi)_j - 1}.$$

If $w_f \in Y \setminus W(T)_{\lambda, \text{ff-inv}}^F$, then the two possibilities in the definition of H' are the same, so $|H'| = 2^{\sum m_\xi(\nu_\xi)_j - 1}$ and the index is $2^{\ell_{w_f}^1 - 1}$. If $w_f \notin Y$, then H' is twice as large (bearing in mind (4.3)), so the index is $2^{\ell_{w_f}^1 - 2}$. Combining this with Proposition 4.2, we have the result. \square

To complete the interpretation of (4.1), we define, for each $w \in Y$, the sign

$$\Phi(w) := \prod_{(\xi, j) \in \Lambda_w^1} (-1)^{\frac{1}{2} m_\xi(\nu_\xi)_j} \prod_{(\xi, j) \in \Lambda_w^2} d_\xi^{\frac{1}{2} m_\xi(\nu_\xi)_j} \prod_{(\xi, j) \in \widetilde{\Lambda}_w^3} d_\xi^{m_\xi(\nu_\xi)_j}.$$

Proposition 4.5. *Let $\bar{f} \in \Theta_{\overline{T}}^F$ be such that $w_f \in W(T)_{\lambda, \text{inv}}^F$.*

- (1) $\mathbb{F}_q\text{-rank}(\overline{T}) + \mathbb{F}_q\text{-rank}(Z_{\overline{G}}((\overline{T} \cap \bar{f} \overline{K} \bar{f}^{-1})^\circ)) \equiv \ell_{w_f}^1 \pmod{2}.$
- (2) $\bar{f} \in \Theta_{\overline{T}, \lambda}^F$ if and only if **either** $w_f \in X \setminus Y$ **or** $w_f \in X \cap Y$ and $\Phi(w_f) = \Phi(\underline{\nu})$.

Proof. The \mathbb{F}_q -rank of \overline{T} is $\ell - 1$, where $\ell = |\Lambda| = \sum_{\xi \in \langle \sigma \rangle \backslash L} \ell(\nu_\xi)$ is the total number of pairs (ξ, j) . Using the description of $(\overline{T} \cap \bar{f} \overline{K} \bar{f}^{-1})^\circ$ given after (4.2) we see that

$$(4.5) \quad Z_{\overline{G}}((\overline{T} \cap \bar{f} \overline{K} \bar{f}^{-1})^\circ) = (GL(\bigoplus_{\substack{(\xi, j, i) \\ (\xi, j) \in \Lambda_{w_f}^1}} L_{(\xi, j, i)}) \times \prod_{\substack{(\xi, j, i) \\ (\xi, j) \notin \Lambda_{w_f}^1}} GL(L_{(\xi, j, i)}))/Z,$$

which has \mathbb{F}_q -rank

$$\sum_{(\xi, j) \in \Lambda_{w_f}^1} m_\xi(\nu_\xi)_j + \ell - \ell_{w_f}^1 - 1,$$

and (1) follows using (4.3). To prove (2), recall that by definition $\bar{f} \in \Theta_{T,\lambda}^F$ iff $\epsilon_{\bar{T},\bar{f}}(\bar{t}) = \lambda(\bar{t})$ for all $\bar{t} \in (\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^F$. Take $t \in T^F$; as in the previous section, its eigenvalues satisfy $\alpha_{(\xi,j,i+1)} = \alpha_{(\xi,j,i)}^q$. By (4.2), $\bar{t} \in (\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^F$ if and only if, for some $\beta \in \mathbb{F}_q^\times$:

- for all $(\xi, j) \in \Lambda_{w_f}^1$, $\alpha_{(\xi,j,1)}^2 = \beta$;
- for all $(\xi, j) \in \Lambda_{w_f}^2$, $\alpha_{(\xi,j,1)}^{1+q^{\frac{1}{2}m_\xi(\nu_\xi)_j}} = \beta$;
- for all $(\xi, j) \in \Lambda_{w_f}^3$, $\alpha_{(\xi,j,1)}^q \alpha_{(\xi,j,1)}^{i(w_f, \xi, j)} = \beta$.

If we let $\pm\sqrt{\beta}$ denote the two square roots of β in $\mathbb{F}_{q^2}^\times$, then

$$(4.6) \quad \begin{aligned} Z_{\bar{G}}^\circ(t) \cap Z_{\bar{G}}((\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ) &= (GL(\bigoplus_{\substack{(\xi,j,i) \\ (\xi,j) \in \Lambda_{w_f}^1 \\ \alpha_{(\xi,j,i)} = \sqrt{\beta}}} L_{(\xi,j,i)}) \times GL(\bigoplus_{\substack{(\xi,j,i) \\ (\xi,j) \in \Lambda_{w_f}^1 \\ \alpha_{(\xi,j,i)} = -\sqrt{\beta}}} L_{(\xi,j,i)})) \\ &\quad \times \prod_{\substack{(\xi,j,i) \\ (\xi,j) \notin \Lambda_{w_f}^1}} GL(L_{(\xi,j,i)}) / Z. \end{aligned}$$

If $\beta \in (\mathbb{F}_q^\times)^2$, then $(\pm\sqrt{\beta})^q = \pm\sqrt{\beta}$, so the \mathbb{F}_q -rank of this group is the same as that of $Z_{\bar{G}}((\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ)$. On the other hand, if $\beta \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$, then $(\pm\sqrt{\beta})^q = \mp\sqrt{\beta}$, so F interchanges the first two GL factors, which means that the \mathbb{F}_q -rank of this group differs from that of $Z_{\bar{G}}((\bar{T} \cap \bar{f}\bar{K}\bar{f}^{-1})^\circ)$ by $\frac{1}{2} \sum_{(\xi,j) \in \Lambda_{w_f}^1} m_\xi(\nu_\xi)_j$; moreover, $m_\xi(\nu_\xi)_j$ must be even for all $(\xi, j) \in \Lambda_{w_f}^1$. Thus

$$(4.7) \quad \epsilon_{\bar{T},\bar{f}}(\bar{t}) = \begin{cases} 1, & \text{if } \beta \in (\mathbb{F}_q^\times)^2, \\ (-1)^{\frac{1}{2} \sum_{(\xi,j) \in \Lambda_{w_f}^1} m_\xi(\nu_\xi)_j}, & \text{if } \beta \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2, \end{cases}$$

the second case being possible only when $w_f \in Y$. Now consider $\lambda(\bar{t}) = \lambda(t)$. If $\beta \in (\mathbb{F}_q^\times)^2$, then

$$\begin{aligned} \lambda(t) &= \prod_{(\xi,j) \in \Lambda_{w_f}^1} \langle \alpha_{(\xi,j,1)}, N(\xi) \rangle^{(\nu_\xi)_j} \prod_{(\xi,j) \in \Lambda_{w_f}^2} \langle \beta, N(\xi) \rangle^{\frac{1}{2}(\nu_\xi)_j} \prod_{(\xi,j) \in \widetilde{\Lambda_{w_f}^3}} \langle \beta, N(\xi) \rangle^{(\nu_\xi)_j} \\ &= \prod_{(\xi,j) \in \Lambda_{w_f}^1} \langle \alpha_{(\xi,j,1)} \sqrt{\beta}^{-1}, N(\xi) \rangle^{(\nu_\xi)_j}, \end{aligned}$$

where the first equality is by the reasoning before Proposition 3.5, and the second is by dividing by $\langle \sqrt{\beta}, \Pi(\underline{\nu}) \rangle = 1$. Since the elements $\alpha_{(\xi,j,1)} \sqrt{\beta}^{-1}$ for $(\xi, j) \in \Lambda_{w_f}^1$ are all ± 1 , and any choice of signs is possible, we see that $\lambda(\bar{t}) = 1$ for all such \bar{t} if and only if $w_f \in X$. So it only remains to consider the case when $w_f \in X \cap Y$ and \bar{t} is such that $\beta \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$. Since $\langle -1, \xi \rangle_{m_\xi(\nu_\xi)_j} = 1$ for all $(\xi, j) \in \Lambda_{w_f}^1$, we have

$$\begin{aligned} \lambda(t) &= \prod_{(\xi,j) \in \Lambda_{w_f}^1} \langle \sqrt{\beta}, \xi \rangle_{m_\xi(\nu_\xi)_j} \prod_{(\xi,j) \in \Lambda_{w_f}^2} \langle \beta^{\frac{1}{2}(\nu_\xi)_j}, \xi \rangle_{m_\xi} \prod_{(\xi,j) \in \widetilde{\Lambda_{w_f}^3}} \langle \beta^{(\nu_\xi)_j}, \xi \rangle_{m_\xi} \\ &= \Phi(\underline{\nu}) \prod_{(\xi,j) \in \Lambda_{w_f}^2} d_\xi^{\frac{1}{2}m_\xi(\nu_\xi)_j} \prod_{(\xi,j) \in \widetilde{\Lambda_{w_f}^3}} d_\xi^{m_\xi(\nu_\xi)_j}, \end{aligned}$$

where we have used the fact that $\langle \beta, \xi \rangle_{m_\xi} = d_\xi^{m_\xi} \langle \sqrt{\beta}, \xi \rangle_{2m_\xi}$. The result follows. \square

Combining (4.1), Lemma 4.3, Proposition 4.4, and Proposition 4.5, we deduce that

$$\begin{aligned} \langle B_{\underline{\nu}}, \text{Ind}_{\overline{K}^F}^{\overline{G}^F}(1) \rangle &= \frac{1}{2}(1 + \epsilon_{\underline{\nu}}) \sum_{w \in W(T)_{\lambda, \text{ff-inv}}^F} \left\{ \begin{array}{ll} 1 & \text{if } \Phi(w) = \Phi(\underline{\nu}) \\ 0 & \text{if } \Phi(w) = -\Phi(\underline{\nu}) \end{array} \right\} \\ &+ \sum_{w \in (X \cap Y) \setminus W(T)_{\lambda, \text{ff-inv}}^F} 2^{\ell_w^1 - 1} (-1)^{\ell_w^1} \left\{ \begin{array}{ll} 1 & \text{if } \Phi(w) = \Phi(\underline{\nu}) \\ 0 & \text{if } \Phi(w) = -\Phi(\underline{\nu}) \end{array} \right\} \\ &+ \sum_{w \in X \setminus Y} 2^{\ell_w^1 - 2} (-1)^{\ell_w^1}. \end{aligned}$$

After some slight rearrangement, and noting that for $w \in W(T)_{\lambda, \text{ff-inv}}^F$ we have $\Phi(w)\Phi(\underline{\nu}) = \langle \beta, \prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{1}{2}|\nu_\xi|} \rangle$, this becomes

$$\begin{aligned} \langle B_{\underline{\nu}}, \text{Ind}_{\overline{K}^F}^{\overline{G}^F}(1) \rangle &= \frac{1}{4} \sum_{w \in X} (-2)^{\ell_w^1} \\ (4.8) \quad &+ \left\{ \begin{array}{ll} \frac{1}{2} \epsilon_{\underline{\nu}} |W(T)_{\lambda, \text{ff-inv}}^F|, & \text{if all } |\nu_\xi| \text{ are even and} \\ & \prod_{\xi \in \langle \sigma \rangle \setminus L} N(\xi)^{\frac{1}{2}|\nu_\xi|} = 1, \\ 0, & \text{otherwise} \end{array} \right. \\ &+ \frac{\Phi(\underline{\nu})}{4} \sum_{w \in X \cap Y} \Phi(w) (-2)^{\ell_w^1}. \end{aligned}$$

The final step is to use the bijection $W(T)_{\lambda, \text{inv}}^F \xrightarrow{\sim} \prod_{\xi \in \langle \sigma \rangle \setminus L} Z_{\text{inv}}^{\nu_\xi}$ to identify the three terms of (4.8) with the three terms of Theorem 2.4.

The first has essentially already been done: in [2, §4.1] it is observed that

$$(4.9) \quad \sum_{w \in X} (-2)^{\ell_w^1} = \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1}} \left(\sum_{w_\xi \in Z_{\text{inv}}^{\nu_\xi}} (-2)^{\ell_{w_\xi}^1(\nu_\xi)} \right) \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = -1}} \left(\sum_{\substack{w_\xi \in Z_{\text{inv}}^{\nu_\xi} \\ \ell_{w_\xi}^1(\nu_\xi)_1 = 0}} (-2)^{\ell_{w_\xi}^1(\nu_\xi)} \right),$$

where $\ell_w^1(\nu)$ and $\ell_w^1(\nu)_1$ are defined as in [2, §1.4]. Hence we need only invoke the combinatorial facts:

$$(4.10) \quad \sum_{w \in Z_{\text{inv}}^{\nu}} (-2)^{\ell_w^1(\nu)} = (-1)^{|\nu|} \sum_{\rho \vdash |\nu|} \left(\prod_i (m_i(\rho) + 1) \right) \chi_\nu^\rho, \text{ and}$$

$$(4.11) \quad \sum_{\substack{w \in Z_{\text{inv}}^{\nu} \\ \ell_w^1(\nu)_1 = 0}} (-2)^{\ell_w^1(\nu)} = \sum_{\substack{\rho \vdash |\nu| \\ \rho' \text{ even}}} \chi_\nu^\rho,$$

which are [2, (4.1.2) and (4.1.3)].

The second term of (4.8) has also been done: in [2, §4.2] it is observed that

$$(4.12) \quad \epsilon_{\underline{\nu}} |W(T)_{\lambda, \text{ff-inv}}^F| = \prod_{\xi \in \langle \sigma \rangle \setminus L} \epsilon_{\nu_\xi} |Z_{\text{ff-inv}}^{\nu_\xi}|,$$

and using (3.5) we obtain the second term of Theorem 2.4.

It only remains to analyse the third term of (4.8). It is clear that under the bijection $W(T)_{\lambda, \text{inv}}^F \xrightarrow{\sim} \prod_{\xi \in \langle \sigma \rangle \setminus L} Z_{\text{inv}}^{\nu_\xi}$, the subset $X \cap Y$ of $W(T)_{\lambda, \text{inv}}^F$ corresponds to the set

$$\{(w_\xi) \in Z_{\text{inv}}^{\nu_\xi} \mid \ell_{w_\xi}^1(\nu_\xi)_1 = 0 \text{ whenever } 2 \nmid m_\xi \text{ or } d_\xi = -1\}.$$

Therefore

$$\begin{aligned}
 \sum_{w \in X \cap Y} \Phi(w) (-2)^{\ell_w^1} &= \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1 \\ 2 \nmid m_\xi}} \sum_{\substack{w_\xi \in Z_{\text{inv}}^{\nu_\xi} \\ \ell_{w_\xi}^1(\nu_\xi)_1 = 0}} (-1)^{\ell_{w_\xi}^1(\nu_\xi)_{2 \bmod 4}} (-2)^{\ell_{w_\xi}^1(\nu_\xi)} \\
 &\quad \times \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = 1 \\ 2 \mid m_\xi}} (-1)^{\frac{1}{2} m_\xi |\nu_\xi|} \sum_{w_\xi \in Z_{\text{inv}}^{\nu_\xi}} (-2)^{\ell_{w_\xi}^1(\nu_\xi)} \\
 &\quad \times \prod_{\substack{\xi \in \langle \sigma \rangle \setminus L \\ d_\xi = -1}} (-1)^{\frac{1}{2} m_\xi |\nu_\xi|} \sum_{\substack{w_\xi \in Z_{\text{inv}}^{\nu_\xi} \\ \ell_{w_\xi}^1(\nu_\xi)_1 = 0}} (-2)^{\ell_{w_\xi}^1(\nu_\xi)}.
 \end{aligned}
 \tag{4.13}$$

In the first of the three groups of factors on the right-hand side, we have used the fact that when m_ξ is odd and $\ell_{w_\xi}^1(\nu_\xi)_1 = 0$,

$$\frac{1}{2} \sum_{j \in \Lambda_{w_\xi}^1(\nu_\xi)} m_\xi(\nu_\xi)_j \equiv \sum_{j \in \Lambda_{w_\xi}^1(\nu_\xi)} \frac{1}{2}(\nu_\xi)_j \equiv \ell_{w_\xi}^1(\nu_\xi)_{2 \bmod 4} \pmod{2},$$

and in the second group of factors we have used the fact that when m_ξ is even,

$$\frac{1}{2} \sum_{j \in \Lambda_{w_\xi}^1(\nu_\xi)} m_\xi(\nu_\xi)_j \equiv \frac{1}{2} m_\xi |\nu_\xi| \pmod{2}.$$

To finish the proof of Theorem 2.4, we apply (4.10) to the second group of factors, (4.11) to the third group of factors, and the following identity to the first group of factors:

$$\sum_{\substack{w \in Z_{\text{inv}}^{\nu} \\ \ell_w^1(\nu)_1 = 0}} (-1)^{\ell_w^1(\nu)_{2 \bmod 4}} (-2)^{\ell_w^1(\nu)} = \sum_{\substack{\rho \vdash |\nu| \\ 2 \mid m_{2i+1}(\rho), \forall i}} (-1)^{\frac{1}{2} |\rho| + \ell(\rho)_{2 \bmod 4}} \left(\prod_i (m_{2i}(\rho) + 1) \right) \chi_\nu^\rho.
 \tag{4.14}$$

This is merely [2, (4.3.2)] with both sides multiplied by $(-1)^{\frac{1}{2} |\nu|}$.

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